

Knowledge and common knowledge

Slides 1 – Common knowledge

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① Epistemic models

Knowledge

Beliefs

② Common knowledge

Almost common knowledge

③ Agreement theorems

No-trade theorems

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① Epistemic models

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Epistemic logic

- Sentences in classical logic are ontological – “*A is true*”
- Ordinary language includes other modalities such as: “*A ought to be true*”, “*A has been true*”, “*A will be true*”, “*A is possible*”, “*A is believed to be true*”, “*A is necessarily true*”
- We are interested in the epistemic modality
- KA means “*A is known to be true*”

S5 axioms for the epistemic operator K

- ① $K(A)$ whenever A is a tautology *(logical omniscience)*
- ② $(A \Rightarrow B) \Rightarrow (KA \Rightarrow KB)$ *(consequence)*
- ③ $KA \Rightarrow A$ *(truth)*
- ④ $KA \Rightarrow KKA$ *(positive introspection)*
- ⑤ $\neg KA \Rightarrow K(\neg KA)$ *(negative introspection)*

Unawareness

“Reports that say that something hasn't happened are always interesting to me, because as we know, there are known knowns; there are things we know we know. We also know there are known unknowns; that is to say we know there are some things we do not know. But there are also unknown unknowns – the ones we don't know we don't know. And if one looks throughout the history of our country and other free countries, it is the latter category that tend to be the difficult ones.”

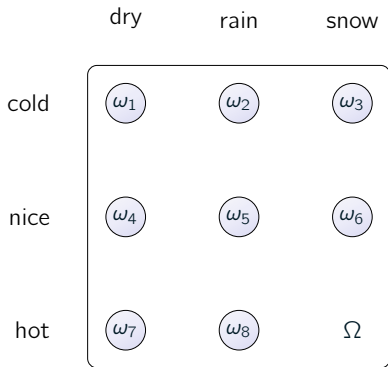
– Donald Rumsfeld

Kripke's semantics

- Epistemic model of the world consisting of:
 - ① Set of conceivable worlds or *states of the world* $\Omega = \{\omega, \omega', \omega'', \dots\}$
An *event* is a set of states of the world $E \in 2^\Omega$
 - ② Accessibility or *possibility* relation \rightarrow on Ω
 $\omega \rightarrow \omega'$ reads “the state of the world ω' is consistent with the information available at ω ”
 - ③ *Knowledge* correspondence $K : 2^\Omega \rightarrow 2^\Omega$
 $K(E)$ is the set of states of the world in which E is known to be true

Example: A simple epistemic model

State space



We are interested in the weather: precipitations and temperature

Possibility

- We require \rightarrow to satisfy for all $\omega, \omega', \omega'' \in \Omega$:

(A1) $\omega \rightarrow \omega$ (*reflexivity*)

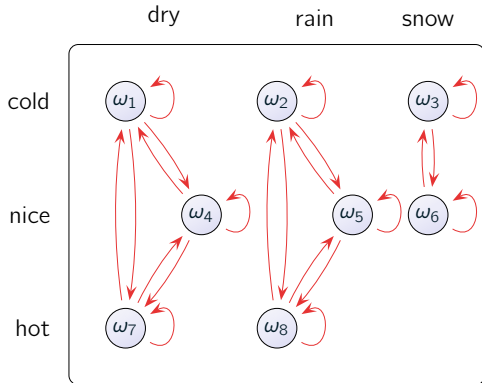
(A2) $\omega \rightarrow \omega' \Rightarrow \omega' \rightarrow \omega$ (*symmetry*)

(A3) $\omega \rightarrow \omega' \rightarrow \omega'' \Rightarrow \omega \rightarrow \omega''$ (*transitivity*)

- i.e. it must be an equivalence relation

Example: A simple epistemic model

Possibility relation



Through the window we can see precipitations but not temperatures

Information partitions

- Possibility can also be described by a *information correspondence* $\pi : \Omega \rightarrow 2^\Omega$:

$$\pi(\omega) = \{\omega' \in \Omega \mid \omega \rightarrow \omega'\}$$

$$\omega \rightarrow \omega' \iff \omega' \in \pi(\omega)$$

- We require that for all $\omega, \omega' \in \Omega$:

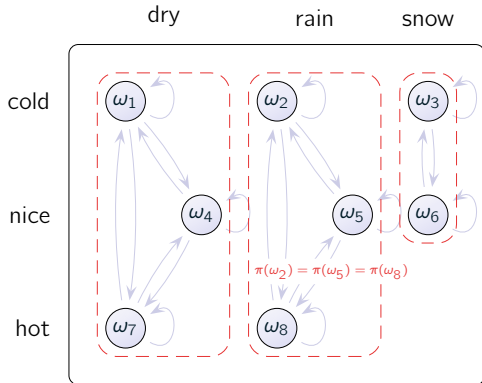
(P1) $\omega \in \pi(\omega)$ for every $\omega \in \Omega$

(P2) If $\omega' \in \pi(\omega)$ then $\pi(\omega) = \pi(\omega')$

- π satisfies (P1)–(P2) if and only if $\Pi \equiv \{\pi(\omega) \mid \omega \in \Omega\}$ is a partition of Ω (lemma 6.8 in the textbook)

Example: A simple epistemic model

Information partition



Through the window we can see precipitations but not temperatures

Information partitions and possibility

Proposition

→ satisfies (A1)–(A3) if and only if π satisfies (P1)–(P2)

- (A1)–(A3) \Rightarrow (P1)–(P2)
 - Reflexivity implies that $\omega \in \pi(\omega)$
 - If $\omega' \in \pi(\omega)$ and $\omega'' \in \pi(\omega')$, then $\omega \rightarrow \omega'$ and $\omega' \rightarrow \omega''$. By transitivity, $\omega \rightarrow \omega''$. Hence $\omega' \in \pi(\omega) \Rightarrow \pi(\omega') \subseteq \pi(\omega)$
 - If $\omega' \in \pi(\omega)$, by reflexivity $\omega \in \pi(\omega')$, and hence $\pi(\omega) \subseteq \pi(\omega')$

Information partitions and possibility

Proposition

\rightarrow satisfies (A1)–(A3) if and only if π satisfies (P1)–(P2)

- (A1)–(A3) \Rightarrow (P1)–(P2)
- (P1)–(P2) \Rightarrow (A1)–(A3)
 - Reflexivity follows from (P1)
 - Suppose $\omega \rightarrow \omega'$. (P2) implies that $\pi(\omega) = \pi(\omega')$, and (P1) implies that $\omega \in \pi(\omega)$. Hence $\omega' \rightarrow \omega$.
 - Suppose $\omega \rightarrow \omega' \rightarrow \omega''$. (P2) implies that $\pi(\omega) = \pi(\omega') = \pi(\omega'')$, and (P1) implies that $\omega'' \in \pi(\omega'')$. Hence $\omega \rightarrow \omega''$.

Knowledge correspondence

- The analogue of the S5 axioms require that for all $E, F \subseteq \Omega$:

$$(K1) \quad K(\Omega) = \Omega \quad (\textit{completeness})$$

$$(K2) \quad K(E) \cap K(F) = K(E \cap F) \quad (\textit{monotonicity})$$

$$(K3) \quad K(E) \subseteq E \quad (\textit{truth})$$

$$(K4) \quad K(E) \subseteq K(K(E)) \quad (\textit{positive introspection})$$

$$(K5) \quad \Omega \setminus K(E) \subseteq K(\Omega \setminus K(E)) \quad (\textit{negative introspection})$$

Knowledge and possibility

- Knowing E is equivalent to knowing that $\Omega \setminus E$ is impossible, hence knowledge can be defined from possibility

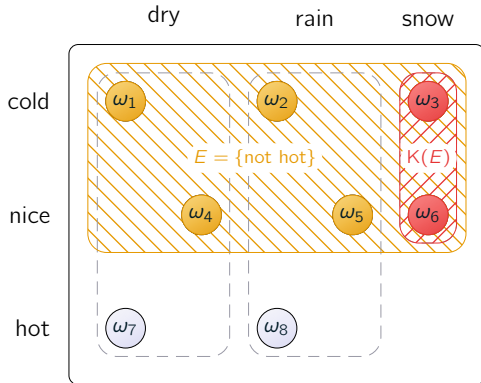
$$K(E) = \{\omega \in \Omega \mid \pi(\omega) \subseteq E\}$$

- Equivalently, a state is thought of as possible if it is not known to be false

$$\pi(\omega) = \{\omega' \in \Omega \mid \omega \notin K(\Omega \setminus \{\omega'\})\}$$

Example: A simple epistemic model

Knowledge



When do we know that it is not hot? Only when it is snowing

Duality I

Proposition

If we define K from π , then $\omega \notin K(\Omega \setminus \{\omega'\})$ if and only if $\omega' \in \pi(\omega)$

- We will show that $\omega \in K(\Omega \setminus \{\omega'\}) \Leftrightarrow \omega' \notin \pi(\omega)$
- Suppose that $\omega \in K(\Omega \setminus \{\omega'\})$
- By construction this happens if and only if $\pi(\omega) \subseteq \Omega \setminus \{\omega'\}$
- Taking complements this happens if and only if $\Omega \setminus \pi(\omega) \supseteq \{\omega'\}$
- Which is equivalent to $\omega' \notin \pi(\omega)$

Duality II

Proposition

If we define π from K and K satisfies monotonicity, then $\pi(\omega) \subseteq E$ if and only if $\omega \in K(E)$

- $\omega \in K(E) \Rightarrow \pi(\omega) \subseteq E$
 - Monotonicity implies $K(E) \subseteq K(\Omega \setminus \{\omega'\})$ for $\omega' \in \Omega \setminus E$
 - Hence if $\omega \in K(E)$ and $\omega' \in \Omega \setminus E$, then $\omega' \notin \pi(\omega)$
 - Which means $\pi(\omega) \subseteq E$

Duality II

Proposition

If we define π from K and K satisfies monotonicity, then $\pi(\omega) \subseteq E$ if and only if $\omega \in K(E)$

- $\omega \in K(E) \Rightarrow \pi(\omega) \subseteq E$
- $\pi(\omega) \subseteq E \Rightarrow \omega \in K(E)$
 - Let $H = \bigcap \{E \subseteq \Omega \mid \omega \in K(E)\}$, monotonicity implies $\omega \in K(H)$
 - For $\omega' \notin \pi(\omega)$, $\omega \in K(\Omega \setminus \{\omega'\})$ and hence $H \subseteq \Omega \setminus \{\omega'\}$
 - Therefore if $\pi(\omega) \subseteq E$ then $H \subseteq \pi(\omega) \subseteq E$ and monotonicity implies that $\omega \in K(E)$

Duality III

Proposition

K satisfies (K1)–(K5) if and only if π satisfies (P1)–(P2)

- If π satisfies (P1)–(P2) then K satisfies (K1)–(K5)

K1 For every $\omega \in \Omega$ we have $\pi(\omega) \subseteq \Omega$

K2 $(\pi(\omega) \subseteq E \wedge \pi(\omega) \subseteq F) \Leftrightarrow \pi(\omega) \subseteq E \cap F$

K3 If $\omega \in K(E)$ then P1 implies that $\omega \in \pi(\omega) \subseteq E$

K4 Let $\omega \in K(E)$

- For $\omega' \in \pi(\omega)$, P2 implies $\pi(\omega') = \pi(\omega) \subseteq E$, and thus $\omega' \in K(E)$
- This means that $\pi(\omega) \subseteq K(E)$, i.e. $\omega \in K(K(E))$

Duality III

Proposition

K satisfies (K1)–(K5) if and only if π satisfies (P1)–(P2)

- If π satisfies (P1)–(P2) then K satisfies (K1)–(K5)

K1 For every $\omega \in \Omega$ we have $\pi(\omega) \subseteq \Omega$

K2 $(\pi(\omega) \subseteq E \wedge \pi(\omega) \subseteq F) \Leftrightarrow \pi(\omega) \subseteq E \cap F$

K3 If $\omega \in K(E)$, then P1 implies that $\omega \in \pi(\omega) \subseteq E$

K4 If $\omega \in K(E)$, then P2 implies $\pi(\pi(\omega)) = \pi(\omega) \subseteq E$ and thus $\pi(\omega) \subseteq K(E)$

K5 Let $\omega \notin K(E)$

- For $\omega' \in \pi(\omega)$, P2 implies $\pi(\omega') = \pi(\omega) \not\subseteq E$, and thus $\omega' \in \Omega \setminus K(E)$
- This means that $\pi(\omega) \subseteq \Omega \setminus K(E)$, i.e. $\omega \in K(\Omega \setminus K(E))$

Duality III

Proposition

K satisfies (K1)–(K5) if and only if π satisfies (P1)–(P2)

- If π satisfies (P1)–(P2) then K satisfies (K1)–(K5)
- If K satisfies (K1)–(K5) then \rightarrow satisfies (A1)–(A3)

A1 From the truth axiom $\omega \notin \Omega \setminus \{\omega\} \supset K(\Omega \setminus \{\omega\})$, and thus $\omega \rightarrow \omega$

A2 Fix $\omega, \omega' \in \Omega$ such that $\omega \not\rightarrow \omega'$

- $\omega \not\rightarrow \omega'$ implies $\{\omega\} \subseteq K(\Omega \setminus \{\omega'\})$
- Taking complements yields $\Omega \setminus \{\omega\} \supseteq \Omega \setminus K(\Omega \setminus \{\omega'\})$
- Monotonicity implies that $K(\Omega \setminus \{\omega\}) \supseteq K(\Omega \setminus K(\Omega \setminus \{\omega'\}))$
- Negative introspection implies that $K(\Omega \setminus \{\omega\}) \supseteq \Omega \setminus K(\Omega \setminus \{\omega'\})$
- A1 then implies that $\omega' \in K(\Omega \setminus \{\omega\})$, i.e. $\omega' \rightarrow \omega$

Duality III

Proposition

K satisfies (K1)–(K5) if and only if π satisfies (P1)–(P2)

- If π satisfies (P1)–(P2) then K satisfies (K1)–(K5)
- If K satisfies (K1)–(K5) then \rightarrow satisfies (A1)–(A3)

A1 From the truth axiom $\omega \notin \Omega \setminus \{\omega\} \supset K(\Omega \setminus \{\omega\})$, and thus $\omega \rightarrow \omega$

A2 From monotonicity and negative introspection \rightarrow is symmetric

A3 Fix $\omega, \omega', \omega'' \in \Omega$ such that $\omega \rightarrow \omega'$ and $\omega' \rightarrow \omega''$

- $\omega' \rightarrow \omega''$ implies $\{\omega'\} \subseteq \Omega \setminus K(\Omega \setminus \{\omega''\})$
- Taking complements yields $\Omega \setminus \{\omega'\} \supseteq K(\Omega \setminus \{\omega''\})$
- Monotonicity then implies $K(\Omega \setminus \{\omega'\}) \supseteq K(K(\Omega \setminus \{\omega''\}))$
- Taking complements again $\Omega \setminus K(\Omega \setminus \{\omega'\}) \subseteq \Omega \setminus K(K(\Omega \setminus \{\omega''\}))$
- Since $\omega \notin K(\Omega \setminus \{\omega'\})$, then $\omega \notin K(K(\Omega \setminus \{\omega''\}))$
- By positive introspection this implies $\omega \notin K(\Omega \setminus \{\omega''\})$, i.e. $\omega \rightarrow \omega''$

Proposition

K satisfies (K1)–(K5) if and only if π satisfies (P1)–(P2)

- If π satisfies (P1)–(P2) then K satisfies (K1)–(K5)
- If K satisfies (K1)–(K5) then \rightarrow satisfies (A1)–(A3)
- If K satisfies (K1)–(K5) then π satisfies (P1)–(P2)

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① Epistemic models

Knowledge

Beliefs

② Common knowledge

Almost common knowledge

③ Agreement theorems

No-trade theorems

Uncertainty

- When $\#\pi(\omega) > 1$ it is now known which is the *true* state of the world
- When uncertain, rational agents form *subjective probabilistic* beliefs about the true state of the world (Savage, 1954)
- Agents' beliefs are described by prior probabilities $p \in \Delta(\Omega)$
 - B1 $p : 2^\Omega \rightarrow [0, 1]$
 - B2 $p(\Omega) = 1$ and $p(\emptyset) = 0$
 - B3 $p(E) = \sum_{\omega \in E} p(\{\omega\})$ for any $E \subseteq \Omega$
- $p(E)$ indicates how likely/plausible the agent thinks that the true state of the world belong to E

Posterior beliefs

- Distinguish between *prior* and *posterior* beliefs, and sometimes *interim*
- Posterior beliefs depend on the state of the world $q : \Omega \rightarrow \Delta(\Omega)$
- $q(E|\omega)$ is the agent's belief about E at ω
- We require for all $\omega \in \Omega$ and $E \subseteq \Omega$

B4 Properness – beliefs are consistent with information

$$q(\pi(\omega)|\omega) = 1$$

B5 Measurability – agent's know their beliefs

$$\omega \in \pi(\omega') \Rightarrow q(\cdot|\omega) = q(\cdot|\omega')$$

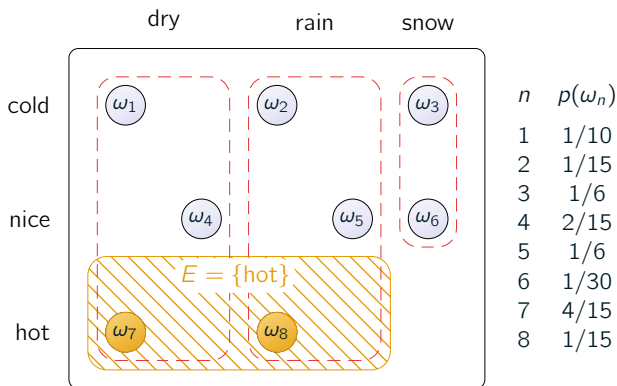
allows to simply define $q : \Pi \rightarrow \Delta(\Omega)$

B6 Bayes consistency, if $p(\pi(\omega)) > 0$ then

$$q(E|\omega) = \frac{p(E \cap \pi(\omega))}{p(\pi(\omega))}$$

Example: A simple epistemic model

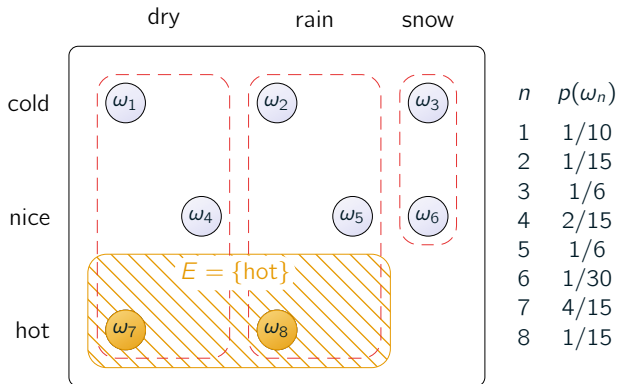
Prior beliefs



The prior probability of hot is $p(E) = \frac{4}{15} + \frac{1}{15} = \frac{1}{3}$

Example: A simple epistemic model

Posterior beliefs

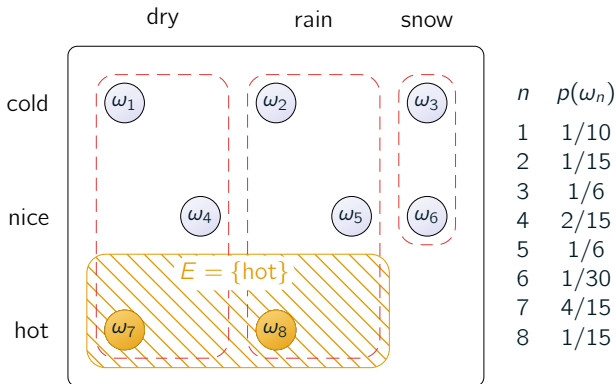


The posterior probability of hot when it's dry is

$$q(E|\omega_1) = q(E|\omega_4) = q(E|\omega_7) = \frac{4/15}{1/10 + 2/15 + 4/15} = \frac{8}{15} > \frac{1}{2}$$

Example: A simple epistemic model

Posterior beliefs



The posterior probability of hot when it's raining is

$$q(E|\omega_1) = q(E|\omega_4) = q(E|\omega_7) = \frac{1/15}{1/15 + 1/6 + 1/15} = \frac{2}{9} < \frac{1}{2}$$

Single agent epistemic model

- Ω – finite set of states of the world
- Π – corresponding information partition satisfying (P1)–(P2)
- K – knowledge operator satisfying (K1)–(K5)
- p – prior beliefs satisfying (B1)–(B3)
- q – posterior beliefs satisfying (B1)–(B6)

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Almost common knowledge

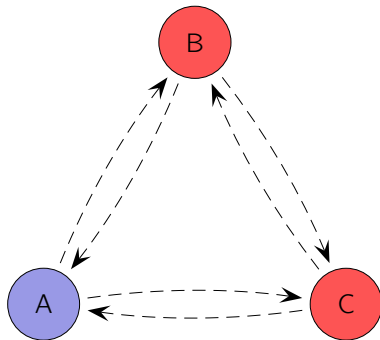
③ Agreement theorems

No-trade theorems

Hierarchies of knowledge and beliefs

- Knowledge and beliefs about what?
 - Incomplete information
payoffs and rules of the game (state of Nature)
 - Imperfect information
choices and beliefs (state of the world)
- Beliefs about beliefs matter
Meeting in NY, Matching pennies, coordinated attack
- **Mutual knowledge** – everyone knows
- **Common knowledge** – everyone knows, everyone knows that everyone knows, everyone knows that everyone knows that everyone knows, . . .

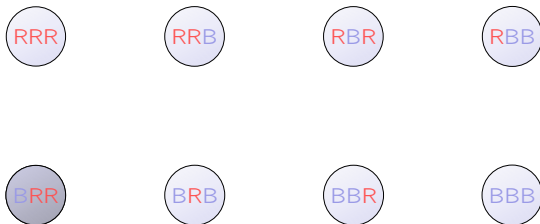
Example: Three hats



- Mutual knowledge of a red hat, but no possible inference
- Common knowledge of a red hat \rightarrow inference becomes possible

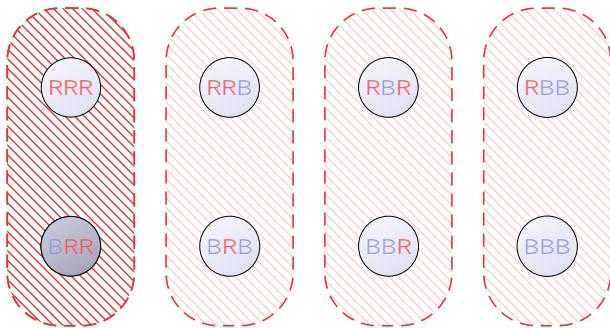
Example: Three hats

States of nature



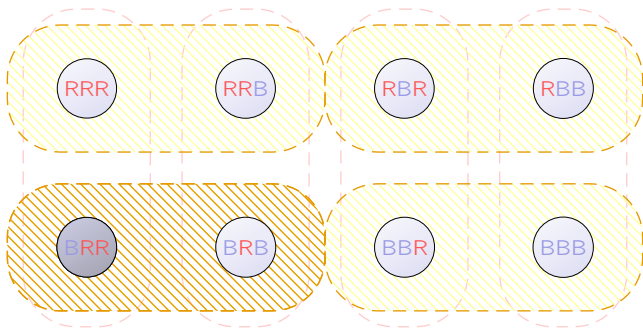
Example: Three hats

Ana's information partition



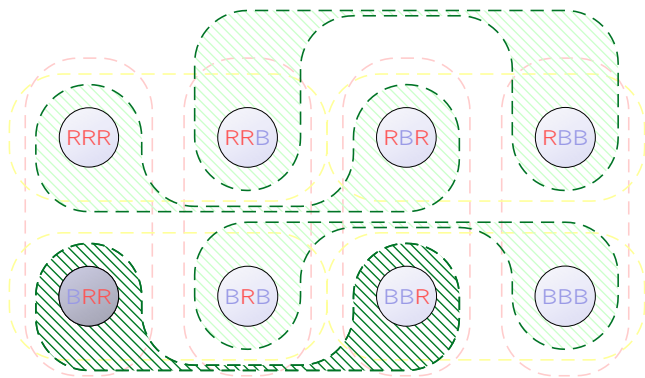
Example: Three hats

Charlie's information partition



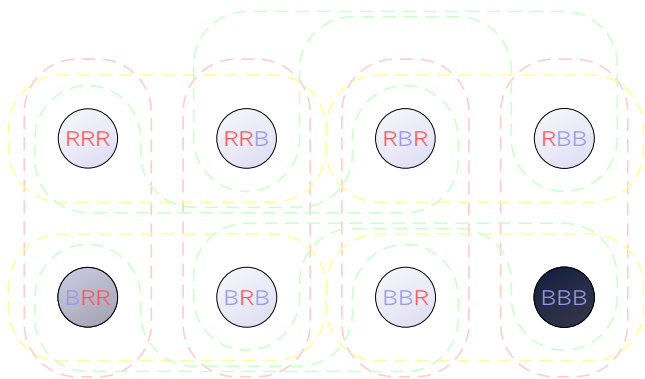
Example: Three hats

Bob's information partition



Example: Three hats

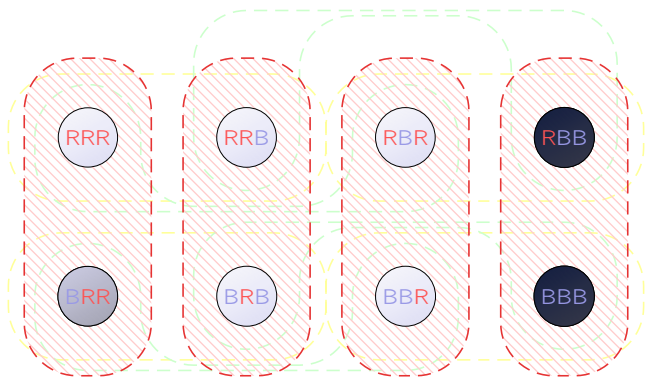
Common knowledge of at least one red hat



After the announcement the state **BBB** is no longer possible

Example: Three hats

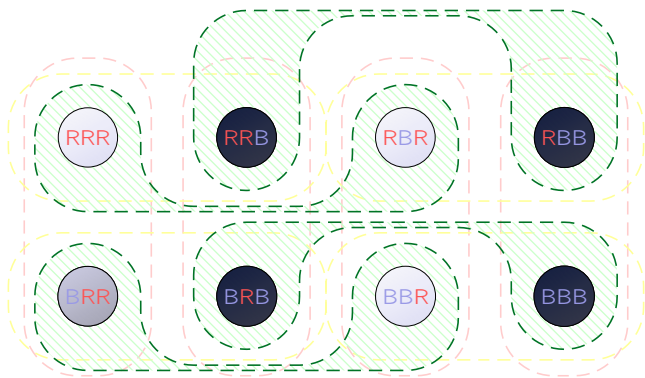
Common knowledge of at least one red hat



Since Anna did not guess the color, **RBB** is not possible

Example: Three hats

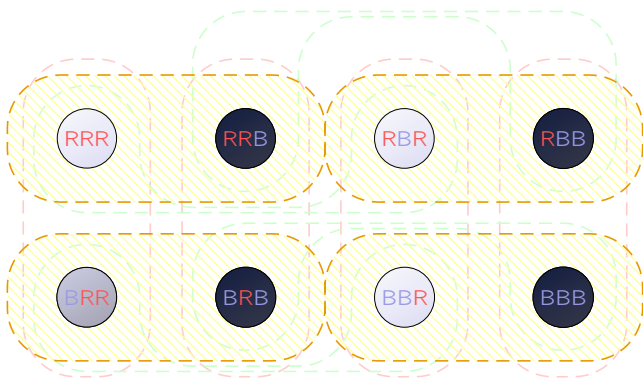
Common knowledge of at least one red hat



Since Bob did not guess the color, RRB and BRB are not possible

Example: Three hats

Common knowledge of at least one red hat



At this point Charlie knows that **BRR** is the only possible state

Also notice that everyone knows that Charlie is the only one who knows the true state

Interactive epistemology

- $I = \{1, 2, \dots, n\}$ – agents
- Ω – states of the world
- Π_i – information partitions
- K_i – corresponding knowledge operators
- p_i – prior beliefs
- q_i – posterior beliefs

- We informally assume that all this information is commonly know, arguably, this comes without loss of generality

Mutual knowledge

- Everyone knows E at ω if $\omega \in K_i(E)$ for all i
- Hence the set of states where E is mutual knowledge is given by:

$$K_I(E) = \bigcap_{i \in I} K_i(E)$$

- Recall $\omega \in K_i(E)$ if and only if $\pi_i(\omega) \subseteq E$, hence:

$$K_I(E) = \{\omega \in \Omega \mid \forall i \in I : \pi_i(\omega) \subseteq E\}$$

Common knowledge

- Everyone knows E at ω if $\omega \in K_1(E)$
- Everyone knows that every one knows E means that everyone knows $K_1(E)$
- Hence everyone knows that everyone knows E at ω if

$$\omega \in K_1^2(E) \equiv K_1(K_1(E))$$

- Every knows that everyone knows that everyone knows E , at ω if

$$\omega \in K_1^3(E) \equiv K_1(K_1^2(E))$$

...

- Therefore E is common knowledge at ω if and only if $\omega \in K_1^n(E)$ for every $n \in \mathbb{N}$

$$K_1^\infty(E) = \bigcap_{n \in \mathbb{N}} K_1^n(E)$$

Self-evident events

- Aumann (1976) provides a useful characterization of CK in terms of self-evident events
- E is **self-evident** for i if *it is known when and only when it is true*
- E is self-evident if *it is mutual knowledge iff it is true*

Proposition

For any event $E \subseteq \Omega$ the following are equivalent:

- ① E is self-evident
- ② $\pi(\omega) \subseteq E$ for all $\omega \in E$
- ③ There exists some $F \subseteq E$ such that $E = \cup_{\omega \in F} \pi(\omega)$

- Self-evident events conform a partition of Ω
- The coarsest partition which refines all the information partitions Π_i

Self-evident events

Proposition

For any event $E \subseteq \Omega$ the following are equivalent:

- ① E is self-evident for i
- ② $\pi_i(\omega) \subseteq E$ for all $\omega \in E$
- ③ $E = \cup_{\omega \in E} \pi_i(\omega)$

- [1 \Rightarrow 2] $\omega \in E \Rightarrow \omega \in K(E) \Rightarrow \pi(\omega) \subseteq E$
- [2 \Rightarrow 3] $\omega \in \pi(\omega)$ for all ω . Hence $E \subseteq \cup_{\omega \in E} \pi_i(\omega)$. From (2) it follows that $E \supseteq \cup_{\omega \in E} \pi_i(\omega)$
- [3 \Rightarrow 1] The truth axiom yields $\omega \in K(E) \Rightarrow \omega \in E$. For $\omega \in E$, (3) implies that $\pi(\omega) \subseteq E$, i.e. $\omega \in K(E)$.

Self-evident events and common knowledge

Proposition

E is common knowledge at ω if and only if ω is contained in a self evident subset of E

- $\omega \in K_1^\infty(E) \Rightarrow (\exists F \subseteq \Omega)(\omega \in F \subseteq E \wedge K_1(F) = F)$
 - $\omega \in K_1^\infty(E) \subseteq \dots \subseteq K_1^2(E) \subseteq K_1(E) \subseteq E$
 - Since 2^Ω is finite and $K_1^n(E)$ is monotone, $K_1^n(E)$ converges in finite time to a non-empty limit F
 - Hence we must have $K_1(F) = F$, i.e. F is self-evident

Self-evident events and common knowledge

Proposition

E is common knowledge at ω if and only if ω is contained in a self evident subset of E

- $\omega \in K_1^\infty(E) \Rightarrow (\exists F \subseteq \Omega)(\omega \in F \subseteq E \wedge K_1(F) = F)$
- $(\exists F \subseteq \Omega)(\omega \in F \subseteq E \wedge K_1(F) = F) \Rightarrow \omega \in K_1^\infty(E)$
 - We will show by induction that $K_1^n(F) = F$ for all $n \in \mathbb{N}$
 - Since F is self evident $K(F) = F$
 - If $F \in K_1^n(F)$, since $K(F) = F$, then $K_1^{n+1}(F) = F$
 - By monotonicity, this implies that $\omega \in K_1^n(F) \subseteq K_1^n(E)$ for all $n \in \mathbb{N}$
 - Hence $\omega \in K_1^\infty(E)$

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① Epistemic models

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Establishing common knowledge

- Achieving common knowledge seems like a difficult task
- Lewis (1969) gives the following account of common knowledge:

... it is common knowledge in a population P that E if and only if some state of affairs A holds such that:

- ① *Everyone in P has reason to believe that A holds.*
- ② *A indicates to everyone in P that everyone in P has reason to believe that A holds.*
- ③ *A indicates to everyone in P that E holds.*

Approximate common knowledge

- Approximate common knowledge is much more plausible
- Different kinds of approximations
 - Common certainty (Brandenburger & Dekel, 1987)

$$C(E) = \bigcap_{i \in I} \{\omega \in \Omega \mid q_i(E|\omega) = 1\} \quad C^\infty(E) = \bigcap_{n \in \mathbb{N}} C^n(E)$$

- p -common belief

$$C_{p_0}(E) = \bigcap_{i \in I} \{\omega \in \Omega \mid q_i(E|\omega) \geq p_0\} \quad C_{p_0}^\infty(E) = \bigcap_{n \in \mathbb{N}} C_{p_0}^n(E)$$

- k -common knowledge $K_1^k(E)$
- If approximations are plausible but exact common knowledge is not, there is a question of robustness or continuity
- Which implications of common knowledge remain when we have approximate common knowledge instead?

Example: electronic mail game (Rubinstein, 1989)

Payoffs

- Two allied armies $i \in \{1, 2\}$ are considering whether to attack a given position of unknown strength
- If only one army attacks, the battle is lost at a cost of $-\theta < 0$ for the attacking army
- If both attack and the position is weak, a victory worth $\gamma \in (0, \theta)$ is won
- If both attack and the position is strong, a loss worth $-\gamma - \theta$ occurs

	A	N
A	γ, γ	$-\theta, 0$
N	$0, -\theta$	$0, 0$

weak

	A	N
A	$-\gamma - \theta, -\gamma - \theta$	$-\theta, 0$
N	$0, -\theta$	$0, 0$

strong

Example: electronic mail game (Rubinstein, 1989)

Information

- Agent 1 is informed about the strength of the position, agent 2 is not
- Agent 2 assigns prior probability $p = 1/2$ to the position being strong
- If the position is weak, agent 1 sends a messenger to alert agent 2
- Upon receiving a message, agent 2 sends the messenger back with a confirmation
- Upon receiving a message, agent 1 sends the messenger back with a confirmation
- ...
- On each travel, there is a probability $\epsilon > 0$ that the messenger is captured and the communication is interrupted

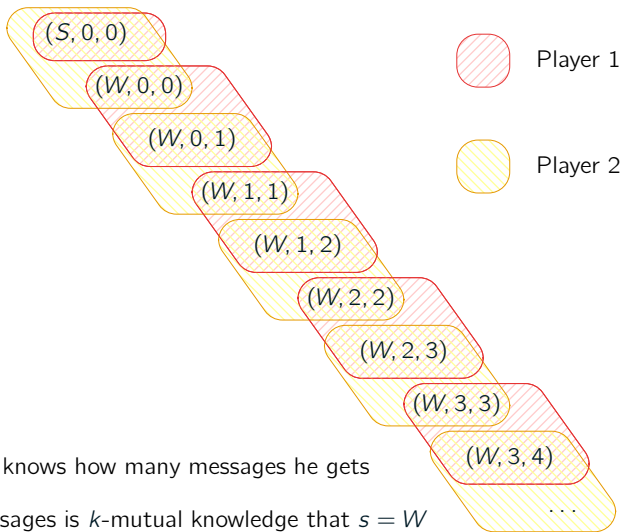
Example: electronic mail game (Rubinstein, 1989)

States of the world

- Let m_i denote the number of messages received by each player
- If the position is strong or it is strong but the first message gets lost, then $m_1 = m_2 = 0$
- If the position is weak and the second message gets lost, then $(m_1, m_2) = (0, 1)$
- If the position is weak and the third message gets lost, then $(m_1, m_2) = (1, 1)$
- If the position is weak and the fourth message gets lost, then $(m_1, m_2) = (1, 2)$
- ...
- A state of the world can be described by (s, m_1, m_2) where wither $s = \text{strong}$ and $m_1 = m_2 = 0$ or $s = \text{weak}$, $m_1 \in \mathbb{Z}_+$ and $m_2 \in \{m_1, m_1 + 1\}$

Example: electronic mail game (Rubinstein, 1989)

Information partitions



Each player knows how many messages he gets

After k messages is k -mutual knowledge that $s = W$

Example: electronic mail game (Rubinstein, 1989)

Optimal choices

- If $s = S$, then player 1 knows that his payoff for attacking is no better than $-\theta$, hence he prefers not to attack
- If $m_2 = 0$, then player 2 believes that $s = S$ with probability:

$$q = \frac{1/2}{1/2 + \varepsilon/2} = \frac{1}{1 + \varepsilon} > \frac{1}{2}$$

hence it is optimal for him to not attack

- If $m_1 = 0$, then player 1 believes that $m_2 = 1$ with probability:

$$q = \frac{\varepsilon}{\varepsilon + \varepsilon^2} = \frac{1}{1 + \varepsilon} > \frac{1}{2}$$

since $\gamma < \theta$, he prefers not to attack, **despite the fact that it is mutual knowledge that the position is weak**

- By induction, players will never attack, independently of how many messages they receive!

[0]

① Epistemic models

Knowledge

Beliefs

② Common knowledge

Almost common knowledge

③ Agreement theorems

No-trade theorems

Common priors

- Harsanyi's doctrine:

"Differences in beliefs are explained by differences in information"

- This assumption can be captured by the common prior assumption:

$$p \equiv p_1 = p_2 = \dots = p_n \quad (\text{CP})$$

- Some justifications for assuming common priors (Morris, 1995):

- ① Perfectly rational agents without any private information have no reason to disagree (Harsanyi, 1967)
- ② Bayesian learning dissipates discrepancies after sufficient common experience (Kalai & Lehrer, 1993)
- ③ Differences in prior beliefs have counterintuitive implications (e.g. arbitrage)
- ④ Assuming common priors is a practical modelling choice

Agreeing to disagree

Theorem (Aumann, 1976)

In an interactive epistemology with common priors, if individual posteriors about an event are common knowledge, then they must be equal.

- Suppose it is common knowledge at ω^* that $q_i(E) = \eta_i$ for $i \in I$
- There exists a self-event F containing ω^* (i.e. $\omega^* \in F = K(F)$) such that $q_i(E|\omega) = \eta_i$ for all $i \in I$ and all $\omega \in F$
- There exists some $H \subseteq F$ such that $F = \cup_{\omega \in H} \pi_i(\omega)$ and $\pi_i(\omega) \cap \pi_i(\omega') = \emptyset$ for $\omega, \omega' \in H$
- It follows that:

$$p(E|F) = \sum_{\omega \in H} p(\pi(\omega)|F)p(E|\pi_i(\omega)) = \sum_{\omega \in H} p(\pi(\omega)|F)q_i(E|\omega) = \eta_i$$

- Therefore for all $i, j \in I$ we have $\eta_i = p(E|F) = \eta_j$

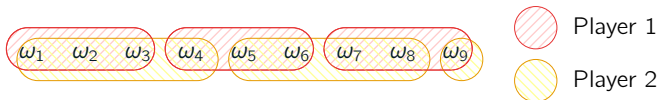
We can't disagree forever

- Aumann's result only applies when posteriors are commonly known, which may be unlikely
- Suppose that posterior beliefs differ (and hence are not common knowledge), but agents discuss them over a sequence of rounds
- In each round, each agent announces her posteriors
- From this announcements, agents can make further inference and revise theirs beliefs

Theorem (Geanakoplos & Polemarchakis, 1982)

In an interactive epistemology with common priors, posteriors will coincide after finitely many rounds of communication.

Example: updating posteriors



- $\Omega = \{\omega_1, \omega_2, \dots, \omega_9\}$, $\omega^* = 1$, $E = \{\omega_3, \omega_4\}$, $p(\omega_n) = 1/9$
- $\Pi_1 = \{\{\omega_1, \omega_2, \omega_3\}, \{\omega_4, \omega_5, \omega_6\}, \{\omega_7, \omega_8, \omega_9\}\}$,
 $\Pi_2 = \{\{\omega_1, \omega_2, \omega_3, \omega_4\}, \{\omega_5, \omega_6, \omega_7, \omega_8\}, \{\omega_9\}\}$
- $q_1(E|\omega^*) = p(\{\omega_3\})/p(\{\omega_1, \omega_2, \omega_3\}) = 1/3$,
 $q_2(E|\omega^*) = p(\{\omega_3, \omega_4\})/p(\{\omega_1, \omega_2, \omega_3, \omega_4\}) = 1/2$
- Posteriors are mutual but not common knowledge, because 2 believes that ω_4 is possible, in which case 1 would believe that ω_5 is possible, in which case we would have $q_2(E|\omega_5) = 0$
- If 2 announces $q_2(E) = 1/2$, and afterwards 1 announces $q_1(E) = 1/3$, player 2 will know that ω_4 is not possible, and then we will have $q_1(E) = q_2(E) = 1/3$

First communication round

- Consider only two players $I = \{1, 2\}$ and suppose (for simplicity) that $p(\omega) > 0$ for all $\omega \in \Omega$
- Let $\omega^* \in \Omega$ be the true state of the world, and let $\pi_i^* = \pi_i(\omega^*)$
- Before any communication, agents beliefs about $E \subseteq \Omega$ at the true state of the world $\omega^* \in \Omega$ are given by:

$$\eta_{i,0} \equiv q_i(E|\omega^*) = \frac{p(E \cap \pi_i^*)}{p(\pi_i^*)}$$

- After the announcements, all players learn that:

$$\omega^* \in B_{i,1} \equiv \left\{ \omega \in \Omega \mid \frac{p(E \cap \pi_i(\omega))}{p(\pi_i(\omega))} = \eta_{i,0} \right\}$$

- The revised beliefs are thus given by:

$$\eta_{i,1} \equiv \frac{p(E \cap \pi_i^* \cap B_{-i,1})}{p(\pi_i^* \cap B_{-i,1})}$$

Subsequent communication rounds

- Players enter the $(t + 1)$ -th period with beliefs:

$$\eta_{i,t} = \frac{p(E \cap \pi_i^* \cap B_{-i,t})}{p(\pi_i^* \cap B_{-i,t})}$$

- After the announcements, all players learn that:

$$\omega^* \in B_{i,t+1} \equiv B_{i,t} \cap \left\{ \omega \in \Omega \mid \frac{p(E \cap \pi_i(\omega) \cap B_{-i,t})}{p(\pi_i(\omega) \cap B_{-i,t})} = \eta_{i,t} \right\}$$

- The revised beliefs are thus given by:

$$\eta_{i,t+1} \equiv \frac{p(E \cap \pi_i^* \cap B_{-i,t})}{p(\pi_i^* \cap B_{-i,t})}$$

Proof that we can't disagree forever

- Since $B_{i,t}$ is a decreasing sequence in a finite set, there exists some $T \in \mathbb{N}$ such that $B_{i,T+1} = B_{i,T}$ for $i \in I$
- Which implies that for $i \in I$ and $\omega \in B_{i,T}$

$$\eta_{i,T} = \frac{p(E \cap \pi_i(\omega) \cap B_{-i,T})}{p(\pi_i(\omega) \cap B_{-i,T})}$$

- Since beliefs are measurable, $B_{i,T}$ is self-evident for i , i.e. there exists some $H_i \subseteq B_{i,T}$ such that $B_{i,T} = \cup_{\omega \in H_i} \pi_i(\omega)$ and $\pi_i(\omega) \cap \pi_i(\omega') = \emptyset$ for $\omega, \omega' \in H_i$
- Hence:

$$\eta_{i,T} \sum_{\omega \in H_i} p(\pi_i(\omega) \cap B_{-i,T}) = \sum_{\omega \in H_i} p(E \cap \pi_i(\omega) \cap B_{-i,T})$$

$$\Rightarrow \eta_{i,T} p(B_{i,T} \cap B_{-i,T}) = p(E \cap B_{i,T} \cap B_{-i,T})$$

$$\Rightarrow \eta_{1,T} = \frac{p(E \cap B_{1,T} \cap B_{-1,T})}{p(B_{1,T} \cap B_{-1,T})} = \eta_{2,T}$$

[0]

① Epistemic models

Knowledge

Beliefs

② Common knowledge

Almost common knowledge

③ Agreement theorems

No-trade theorems

Example: information bets (Sebenius & Geanakoplos, 1983)

- Anna and Bob are risk neutral, i.e. they maximize expected wealth
- They disagree on who is more likely to win the World Cup
- Is it possible for them to make a bet?
- Not if they are rational, and have common priors!
- If Ana is willing to bet on Spain, it means that her information indicates that Spain is more likely to win
- If Bob is willing to accept the bet, it means that his information indicates that Brazil is more likely to win
- Making inferences will lead one of them to update his/her beliefs and reject the bet

Proposition

In an interactive epistemology with common priors, given a random variable \mathbf{x} , it cannot be common knowledge that $\mathbb{E}_j[\mathbf{x}] > 0$ and $\mathbb{E}_j[\mathbf{x}] < 0$

Speculative trading

- A financial asset is essentially a random variable which pays different results depending on the (unknown) state of the world
- Financial assets may be traded for (at least) two reasons
 - Allocation of risk and liquidity (smoothing and insurance)
 - Potential gains from asymmetric information
- The previous result can be generalized to show that *speculative trading is not possible among rational individuals with common priors* (Stokey & Lucas, 1982)

Thanks

This concludes the first part of the course!

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